

Charging black Saturn?

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ABSTRACT: We construct new charged static solutions of the Einstein-Maxwell field equations in five dimensions via a solution generation technique utilizing the symmetries of the reduced Lagrangian. By applying our method on the multi-Reissner-Nordström solution in four dimensions, we generate the multi-Reissner-Nordström solution in five dimensions. We focus on the five-dimensional solution describing a pair of charged black objects with general masses and electric charges. This solution includes the double Reissner-Nordström solution as well as the charged version of the five-dimensional static black Saturn. However, all the black Saturn configurations that we found contain either a conical or naked singularity. We also obtain a non-extremal configuration of charged black strings that reduces in the extremal limit to a Majumdar-Papapetrou like solution in five dimensions.

KEYWORDS: Classical Theories of Gravity, Black Holes.

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1. Introduction

Higher dimensional black hole solutions have been known for a long time, for example the Schwarzschild-Tangherlini black holes, their charged Reissner-Nordström cousins, as well as the higher dimensional generalization of the rotating Kerr solution [1, 2]. In the past few years there has been remarkable progress in this field, notably the discovery of asymptotically flat black holes with non-spherical horizon topology. A particularly interesting case is the five dimensional asymptotically flat black ring solution whose horizon topology is $S^2 \times S^1$ instead of the usual S^3 topology of the Schwarzschild solution [3].

The existence of five-dimensional black rings revealed that certain four-dimensional features of General Relativity cannot be easily extended to dimensions greater than four. For instance, the celebrated ‘no-hair’ theorem of four dimensional black hole physics does not hold in more than four dimensions. According to the theorem, an asymptotically flat, stationary charged black hole is uniquely characterized by its mass, charge and angular momentum and can only have a horizon with spherical topology. This is violated in five dimensions where one can have exact solutions describing black rings with non-spherical horizon topology and at the same time not fully characterized by its conserved charges [4, 5].

In this paper we are interested in five dimensional multi-black hole solutions related to the black ring solution. Using the recent extension of the Weyl formalism to dimensions greater than four [6], the construction of the static five-dimensional multi-black hole

solution was carried out in [7]. One of the major tasks in multi-black hole physics is how to maintain the black holes in equilibrium. It turns out that in the static vacuum case in five dimensions, conical singularities are required to generically induce struts of stress energy to counter their mutual gravitational attraction just as in four dimensions. An alternative to conical singularities is to use rotation to keep the black holes apart. This is apparent in the case of a single five-dimensional black ring where its angular momentum provides the necessary force to keep the black ring from collapsing. This mechanism is also present in the asymptotically flat black Saturn solution in five dimensions, where a black hole in the center of a rotating black ring can be in equilibrium if the black ring rotates fast enough [8, 9]. One other natural candidate for stabilizing a static black ring is a gauge field, in the simplest case an electromagnetic field, and an exact solution describing an electrically charged static black ring was soon found [10–16]. However the presence of an electric charge alone was found insufficient to stabilize the black ring and prevent it from collapsing, since conical singularities in this solution were unavoidable. Nonetheless, by submerging a charged static black ring into an electric/magnetic background field the conical singularities were eliminated and the static black ring stabilized. The only drawback of this construction was that, due to the backreaction of the background electromagnetic field, the black ring was no longer asymptotically flat.

This leads us to conjecture that by introducing a gauge field to counter the gravitational forces in a multi-black hole system its constituents could be kept static, with the electrostatic repulsion between two charged objects counteracting their mutual gravitational attraction. For instance, in four dimensions there exist static configurations of extremal Reissner-Nordström black holes. Similar extremal configurations also exist in higher dimensions [17], the supersymmetric configurations in five dimensions have been constructed in [18, 19] and dynamical solutions exist in lower dimensions [20]. However, the general non-extremal charged multi-black hole solutions are still so far unknown. One may consider a similar situation in the case of a static charged black ring immersed in a background electromagnetic field where the electric field generated by the charged black hole sitting in the center of the static black ring has a stabilizing effect. We thus anticipate the existence of a charged version of the black Saturn in five dimensions, i.e. a charged black ring (non-rotating) kept in equilibrium by the electric field of a charged black hole sitting in its center.

In order to verify the conjecture of the existence of a static charged black Saturn in equilibrium, one has to construct the complete multi-black hole solutions in five dimensional Einstein-Maxwell theory. The main purpose of this paper is to show how one accomplishes this goal. However, for simplicity we shall restrict our attention to configurations consisting of only two constituents. These solutions will include as special cases the charged black Saturn solution, the double non-extremal Reissner-Nordström solution and the double black string solution, whose extremal limit is precisely a string-like variant of the five dimensional Majumdar-Papapetrou solution. We also note that our generated solution can describe configurations of two black rings (orthogonal or concentric). Although a static black Saturn in $d = 5$ Einstein-Maxwell theory has been constructed in recent works [21, 22], this solution is kept in equilibrium by an external magnetic field and approaches at infinity a Melvin

universe background. By contrast, all our generated solutions are asymptotically flat. Unfortunately, we were unable to find non-singular equilibrium black Saturn configurations: we found that either a conical singularity or a naked curvature singularity must be present. The presence of naked curvature singularities is due to the fact that one ‘mass’ parameter is negative - even though the Komar masses of the constituents are positive. The total ADM mass as measured at infinity is also positive.

As is well known, Einstein’s field equations form a set of nonlinear, coupled partial differential equations. Solving them analytically by brute force is a formidable task except in the most simplified cases. However, by considering spacetime geometries endowed with particular symmetries, it is sometimes possible to derive solutions in a systematic way. Some of the most powerful known techniques in constructing exact solutions in General Relativity in higher dimensions require spacetime geometries to be of the generalized Weyl class as described in [6]. In general, one drawback of the generalized Weyl formalism is that it is limited to $D \leq 5$ since general black holes in $D > 5$ dimensions do not admit $(D - 2)$ commuting Killing vectors. For our aim of generating the general charged multi-black hole solution in five dimensions, this limitation does not affect us.

The structure of this paper is as follows. We first describe the solution generating technique that will allow us to lift four-dimensional charged static configurations to five dimensions. We then use the general double Reissner-Nordström solutions in four dimensions as a seed and lift it to five dimensions. We consider in detail the properties of the charged double-black hole solution as well as those of the charged black Saturn configuration. Our solution generating method extends easily to the more general case of Einstein-Maxwell-Dilaton (EMD) gravity with arbitrary coupling constant and we derive the charged multi-black hole solutions in this case. We end with a summary of our work and consider avenues for future research.

2. Solution generating technique

In this section we present a solution generating technique that will map a general static axisymmetric solution of the Einstein-Maxwell theory in four dimensions to a five dimensional static axisymmetric solution of the Einstein-Maxwell-Dilaton (EMD) theory with general dilaton coupling. The solution generating method will allow us to bypass the actual solving of Einstein’s equations as it is based on a comparison of the reduced Lagrangians of the two theories in three dimensions and the mapping of the corresponding scalar fields and electromagnetic potentials. This idea can be traced back to previous work done in four dimensions to relate stationary axisymmetric vacuum solutions to solutions of the EMD system [23]. However, we show that the analogous mapping in our case can be further modified by introducing new harmonic functions in the final solution. This new harmonic ‘degree of freedom’ is essential in the correct construction of the five dimensional solutions.

Our starting point is the five dimensional Lagrangian describing gravity coupled to a dilaton field ϕ and a 2-form field strength $F_{(2)}$:

$$\mathcal{L}_5 = \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{\alpha\phi}F_{(2)}^2 \right], \quad (2.1)$$

where $F_{(2)} = dA_{(1)}$, and the only non-zero component of the 1-form gauge potential $A_{(1)}$ is A_t . We assume that both A_t and the scalar field ϕ depend only on the coordinates ρ and z .

Let us adopt the following axisymmetric metric ansatz in five dimensions and assume that f, k, l and μ depend on the coordinates ρ and z only:

$$ds_5^2 = -f dt^2 + l d\varphi^2 + k d\chi^2 + e^\mu (d\rho^2 + dz^2). \quad (2.2)$$

We now perform a double dimensional reduction down to three dimensions, first along the coordinate χ then along the time coordinate t . Our metric ansatz is:

$$ds_5^2 = e^{\frac{\phi_1}{\sqrt{3}}} [e^{\phi_2} ds_3^2 - e^{-\phi_2} dt^2] + e^{-\frac{2\phi_1}{\sqrt{3}}} d\chi^2,$$

and one obtains:

$$\begin{aligned} ds_3^2 &= e^\mu f k (d\rho^2 + dz^2) + f l k d\varphi^2, \\ e^{-\phi_2} &= f \sqrt{k}, & e^{-\frac{\phi_1}{\sqrt{3}}} &= \sqrt{k}, & A_{(1)} &= A_t dt, \end{aligned} \quad (2.3)$$

which is a solution of the equations of motion derived from the following Lagrangian:

$$\mathcal{L}_3 = \sqrt{g} \left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 + \frac{1}{2}e^{\phi_2 - \frac{\phi_1}{\sqrt{3}} + \alpha\phi} (\partial A_t)^2 \right]. \quad (2.4)$$

We now identify the Lagrangian describing the dynamics of the three dimensional matter fields as:

$$\mathcal{L}_{\text{EMD}}^{\text{matter}} = \sqrt{g} \left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 + \frac{1}{2}e^{\phi_2 - \frac{\phi_1}{\sqrt{3}} + \alpha\phi} (\partial A_t)^2 \right]. \quad (2.5)$$

Consider now the four-dimensional Einstein-Maxwell Lagrangian:

$$\mathcal{L}_4 = \sqrt{-g} \left[R - \frac{1}{4} \tilde{F}_{(2)}^2 \right], \quad (2.6)$$

where $\tilde{F}_{(2)} = d\tilde{A}_{(1)}$ and the only non-zero component of $\tilde{A}_{(1)}$ is $\tilde{A}_t = \omega$. The solution to the equations of motion derived from (2.6) is assumed to have the following static and axisymmetric form:

$$\begin{aligned} ds_4^2 &= -\tilde{f} dt^2 + \tilde{f}^{-1} [e^{2\tilde{\mu}} (d\rho^2 + dz^2) + \rho^2 d\varphi^2], \\ \tilde{A}_{(1)} &= \omega dt. \end{aligned} \quad (2.7)$$

We next perform a Kaluza-Klein reduction along the timelike direction using the metric ansatz:

$$ds_4^2 = e^\psi ds_3^2 - e^{-\psi} dt^2, \quad (2.8)$$

to obtain the following metric and fields in three dimensions:

$$\begin{aligned} ds_3^2 &= e^{2\tilde{\mu}} (d\rho^2 + dz^2) + \rho^2 d\varphi^2, \\ e^{-\psi} &= \tilde{f}, & \tilde{A}_{(0)t} &= \omega, \end{aligned} \quad (2.9)$$

where we have denoted the scalar from Kaluza-Klein reduction by ψ . The above is a solution to the equations of motion derived from the three dimensional Lagrangian:

$$\mathcal{L}_3 = \sqrt{g} \left[R - \frac{1}{2}(\partial\psi)^2 + \frac{1}{2}e^\psi(\partial\omega)^2 \right], \quad (2.10)$$

The Lagrangian describing the dynamics of the three dimensional matter fields is:

$$\mathcal{L}_{\text{EM}}^{\text{matter}} = \sqrt{g} \left[-\frac{1}{2}(\partial\psi)^2 + \frac{1}{2}e^\psi(\partial\omega)^2 \right]. \quad (2.11)$$

In order to relate a solution to the field equations derived from (2.11) to a solution of the field equations derived from (2.5) we shall consider as an intermediary step the following field definitions, starting from a given solution (ψ, ω) of (2.11):¹

$$\bar{\phi} = \frac{3\alpha}{3\alpha^2 + 4}\psi, \quad \bar{\phi}_1 = -\frac{\sqrt{3}}{3\alpha^2 + 4}\psi, \quad \bar{\phi}_2 = \frac{3}{3\alpha^2 + 4}\psi, \quad (2.12)$$

while we also transform the electric 1-form potential as

$$\bar{A}_t = \sqrt{\frac{3}{3\alpha^2 + 4}}\omega. \quad (2.13)$$

One notices then the following relation between the three dimensional reduced matter Lagrangians:

$$\mathcal{L}_{\text{EM}}^{\text{matter}} = \left(\frac{3}{3\alpha^2 + 4} \right) \bar{\mathcal{L}}_{\text{EMD}}^{\text{matter}}. \quad (2.14)$$

Since we have scaled the reduced three-dimensional matter Lagrangian by a constant factor, in order to match the solutions of the equations of motion derived from the above lagrangians we also have to modify their three dimensional geometries such that the Ricci tensor of the metric (2.3) is essentially a constant rescaling of the Ricci tensor of the metric (2.9), the scaling factor being $\frac{3}{3\alpha^2 + 4}$. By comparing the three-dimensional geometries and taking into consideration the special properties of the Weyl-Papapetrou ansatz (2.9) in three dimensions (see [23] for more details) this can be easily accomplished by taking:²

$$e^{\bar{\mu}}\bar{f}\bar{k} \equiv (e^{2\bar{\mu}})^{\frac{3}{3\alpha^2 + 4}}, \quad \bar{f}\bar{k}\bar{l} \equiv \rho^2. \quad (2.15)$$

One can check that the ‘barred’ fields $(\bar{\mu}, \bar{f}, \bar{k}, \bar{l}, \bar{A}_t)$ solve the equations of motion derived from (2.5) as expected. Note that there also exists a freedom in defining the scalar fields (2.12), which can be seen from considering new scalars $\phi = \bar{\phi}$, $\phi_1 = \bar{\phi}_1 - \sqrt{3}h$ and $\phi_2 = \bar{\phi}_2 - h$ such that the new matter Lagrangian can be written as:

$$\begin{aligned} \mathcal{L}_{(1)}^{\text{matter}} &= \sqrt{g} \left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 + \frac{1}{2}e^{\phi_2 - \frac{\phi_1}{\sqrt{3}} + \alpha\phi}(\partial A_t)^2 \right] \\ &= \sqrt{g} \left[-\frac{1}{2}(\partial\bar{\phi})^2 - \frac{1}{2}(\partial\bar{\phi}_1)^2 - \frac{1}{2}(\partial\bar{\phi}_2)^2 + \frac{1}{2}e^{\bar{\phi}_2 - \frac{\bar{\phi}_1}{\sqrt{3}} + \alpha\bar{\phi}}(\partial\bar{A}_t)^2 + 4(\partial h)^2 \right], \end{aligned} \quad (2.16)$$

¹Note that with this choice the equations of motion derived from (2.5) are indeed satisfied, such that (2.11) is a consistent truncation of (2.5).

²This is the scaling symmetry property used in [24] to derive new solutions in four dimensions.

where in the second line of the above equality we used (2.12) to eliminate the cross-terms containing products of the ‘barred’ scalar fields with h . The field h is thus decoupled from the other matter fields.

Notice that in order to obtain a five dimensional solution described by the scalar fields ϕ , ϕ_1 and ϕ_2 the initial ‘barred’ five dimensional Einstein-Maxwell-Dilaton solution should be modified to accommodate the extra scalar field h . Notice further that h must be a harmonic function (as can be seen from its equations of motion) and, moreover, since the ‘barred’ five-dimensional EMD fields are not directly coupled to it, its gravitational backreaction is easily taken care of by introducing a new function γ such that:

$$\partial_\rho \gamma = \rho[(\partial_\rho h)^2 - (\partial_z h)^2], \quad \partial_z \gamma = 2\rho(\partial_\rho h)(\partial_z h). \quad (2.17)$$

Hence, given a harmonic function h , we can solve (2.17) for γ , which can then be substituted in the following metric:

$$ds_5^2 = (e^{\bar{\mu}} \bar{f} \bar{k}) e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\varphi^2, \quad (2.18)$$

to obtain a solution to the modified Lagrangian (2.16).

Taking into account the scaling of the three dimensional metric and presence of the harmonic function h , we have the following relations:

$$e^\mu f k \equiv (e^{2\bar{\mu}})^{\frac{3}{3\alpha^2+4}} e^{2\gamma}, \quad f k l \equiv \rho^2, \quad (2.19)$$

where γ can be found from (2.17) once h is known.

Let us now summarize the results of our solution generating method. One reads off the functions \tilde{f} , ω and $e^{\tilde{\mu}}$ from the four-dimensional metric (2.7) and then substitutes them into the transformations (2.12) and (2.13). The harmonic function h can alter the metric only along the spatial directions and its form is guessed by imposing a desired background geometry in the final solution. Using (2.3) and (2.19), one then computes f, k, l and e^μ in terms of $e^{\tilde{\mu}}$, \tilde{f} , h and γ . The result is then a new EMD solution in five dimensions, which can be written as:

$$ds_5^2 = -\tilde{f}^{\frac{4}{3\alpha^2+4}} dt^2 + \tilde{f}^{-\frac{2}{3\alpha^2+4}} \left[e^{2h} d\chi^2 + e^{\frac{6\tilde{\mu}}{3\alpha^2+4} + 2\gamma - 2h} (d\rho^2 + dz^2) + \rho^2 e^{-2h} d\varphi^2 \right], \quad (2.20)$$

while the 1-form potential and the dilaton are given by:

$$A_{(1)} = \sqrt{\frac{3}{3\alpha^2+4}} \omega dt, \quad e^{-\phi} = \tilde{f}^{\frac{3\alpha}{3\alpha^2+4}}. \quad (2.21)$$

Solutions of the pure Einstein-Maxwell theory in five dimensions are simply obtained from the above formulae by taking $\alpha = 0$. In the following sections we shall focus on this case.

3. Multi-Reissner-Nordström solutions in five dimensions

As a check of the technique presented in the last section, we will first map the four-dimensional Reissner-Nordström solution to the five-dimensional Reissner-Nordström solution. We then use the four-dimensional double-Reissner-Nordström solution in a form recently given by Manko [25] as the seed to generate the double-Reissner-Nordström solution in five dimensions. This four-dimensional solution has been recently re-derived in [26] by using a monodromy transform approach.

3.1 Single Reissner-Nordström black holes and charged black rings in five dimensions

The four-dimensional Reissner-Nordström solution is written in Weyl form as [27]:

$$\begin{aligned}
 ds^2 &= -\tilde{f}dt^2 + \tilde{f}^{-1}[e^{2\tilde{\mu}}(d\rho^2 + dz^2) + \rho^2 d\varphi^2], \\
 \omega &= -\frac{4q}{r_1 + r_2 + 2m}, \quad \tilde{f} = \frac{(r_1 + r_2)^2 - 4\sigma^2}{(r_1 + r_2 + 2m)^2}, \quad e^{2\tilde{\mu}} = \frac{(r_1 + r_2)^2 - 4\sigma^2}{4r_1 r_2},
 \end{aligned}
 \tag{3.1}$$

where

$$r_1 = \sqrt{\rho^2 + (z - \sigma)^2}, \quad r_2 = \sqrt{\rho^2 + (z + \sigma)^2}.
 \tag{3.2}$$

Note that $\sigma = \sqrt{m^2 - q^2}$ and m denotes the mass and q the charge.

The five-dimensional Reissner-Nordström metric is given by (2.20) with $\alpha = 0$ once we use a suitable harmonic function h to ensure that the generated five-dimensional metric is also asymptotically flat. With hindsight, we find that the appropriate h is given by:

$$\begin{aligned}
 e^{2h} &= (r_2 + (z + \sigma)) \left(\frac{r_1 + (z - \sigma)}{r_2 + (z + \sigma)} \right)^{\frac{1}{2}} \\
 &= [(r_2 + \zeta_2)(r_1 + \zeta_1)]^{\frac{1}{2}},
 \end{aligned}
 \tag{3.3}$$

and we can now find γ from (2.17):

$$e^{2\gamma} = \frac{[(r_2 + \zeta_2)(r_1 + \zeta_1)]^{\frac{1}{2}}}{[8r_1 r_2 Y_{12}]^{\frac{1}{4}}}.
 \tag{3.4}$$

where $\zeta_1 = z - \sigma$, $\zeta_2 = z + \sigma$ and $Y_{12} = r_1 r_2 + \zeta_1 \zeta_2 + \rho^2$. The first factor in e^{2h} moves the semi-infinite rod $z < -\sigma$ from the φ direction to the χ direction, while the second factor corresponds to a ‘correction’ of the black hole horizon. It turns out that we will have to take such horizon corrections into account for each horizon when describing multi-black objects in five dimensions, while the rod-moving terms in h can be read from the expected rod structure in the final geometry.

We thus obtain:³

$$\begin{aligned}
 ds_5^2 &= -\frac{(r_1 + r_2)^2 - 4\sigma^2}{(r_1 + r_2 + 2m)^2} dt^2 + \frac{r_1 + r_2 + 2m}{\sqrt{2Y_{12}}} \left[\sqrt{(r_2 + \zeta_2)(r_1 + \zeta_1)} d\chi^2 + \frac{\sqrt{2Y_{12}}}{4r_1 r_2} (d\rho^2 + dz^2) \right. \\
 &\quad \left. + \frac{\rho^2 d\varphi^2}{\sqrt{(r_2 + \zeta_2)(r_1 + \zeta_1)}} \right].
 \end{aligned}
 \tag{3.5}$$

Let us now convert the above metric from cylindrical coordinates (ρ, z) to polar coordinates (r, θ) by using the relations [6]:

$$\rho^2 = r^2(r^2 - 4\sigma) \sin^2 \theta \cos^2 \theta, \quad z = \frac{1}{2}(r^2 - 2\sigma) \cos 2\theta.
 \tag{3.6}$$

³Note that $2Y_{12} = (r_1 + r_2)^2 - 4\sigma^2$.

We obtain:

$$\begin{aligned}
 ds_5^2 &= -\frac{r^2(r^2 - 4\sigma)}{(r^2 + 2(m - \sigma))^2} dt^2 \\
 &\quad + \frac{r^2 + 2(m - \sigma)}{r^2} \left(\frac{r^2}{r^2 - 4\sigma} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\chi^2) \right) \\
 &= -H^{-2}(r) f(r) dt^2 + H(r) (f(r)^{-1} dr^2 + r^2 d\Omega_3^2), \\
 A_t &= -\frac{2\sqrt{3}\sqrt{m^2 - \sigma^2}}{r^2 + 2(m - \sigma)}, \tag{3.7}
 \end{aligned}$$

where

$$H(r) = 1 + \frac{2(m - \sigma)}{r^2}, \quad f(r) = 1 - \frac{4\sigma}{r^2}, \tag{3.8}$$

which is indeed the five-dimensional Reissner-Nordström solution. It should be clear that if we relax the $\alpha = 0$ condition one can obtain from (2.20) the dilatonic black hole found previously in [28].

If one chooses the following harmonic function h instead:

$$\begin{aligned}
 e^{2h} &= (r_0 + (z + \sigma_0)) \left(\frac{r_1 + (z - \sigma)}{r_2 + (z + \sigma)} \right)^{\frac{1}{2}}, \\
 &= (r_0 + \zeta_0) \left(\frac{r_1 + \zeta_1}{r_2 + \zeta_2} \right)^{\frac{1}{2}}, \tag{3.9}
 \end{aligned}$$

one readily sees that the rod structure of the final solution corresponds to a static black ring. Here we denote $r_0 = \sqrt{\rho^2 + (z + \sigma_0)^2}$ and $\zeta_0 = z + \sigma_0$, where $\sigma_0 > \sigma > 0$. We can now find γ from (2.17):

$$e^{2\gamma - 2h} = \frac{1}{K_0 r_0} \left(\frac{Y_{02}}{Y_{01}} \right)^{\frac{1}{2}} \left(\frac{4Y_{12}}{r_1 r_2} \right)^{\frac{1}{4}}, \tag{3.10}$$

where $Y_{ij} = r_i r_j + \zeta_i \zeta_j + \rho^2$, $i, j = 0, 1, 2$ and K_0 is an integration constant. In Weyl coordinates the charged black ring solution is then found to be:

$$\begin{aligned}
 ds^2 &= -\frac{(r_1 + r_2)^2 - 4\sigma^2}{(r_1 + r_2 + 2m)^2} dt^2 \\
 &\quad + \frac{r_1 + r_2 + 2m}{\sqrt{2Y_{12}}} \left[(r_0 + \zeta_0) \sqrt{\frac{r_1 + \zeta_1}{r_2 + \zeta_2}} d\chi^2 + \frac{2Y_{12}}{K_0 r_0 r_1 r_2} \sqrt{\frac{Y_{02}}{Y_{01}}} (d\rho^2 + dz^2) \right. \\
 &\quad \left. + \frac{\rho^2 d\varphi^2}{r_0 + \zeta_0} \sqrt{\frac{r_2 + \zeta_2}{r_1 + \zeta_1}} \right], \\
 A_t &= -\frac{\sqrt{3}}{2} \frac{4\sqrt{m^2 - \sigma^2}}{r_1 + r_2 + 2m}. \tag{3.11}
 \end{aligned}$$

The metric of the uncharged static black ring (see for instance equations (4.15 – 4.18) in [6]) is recovered in the limit $m = \sigma$, thus confirming that the above solution describes a

static black ring in Weyl coordinates. We have thus generated the static charged black ring as a solution of Einstein-Maxwell-Dilaton system in five dimensions, a solution previously found in [12].

3.2 The double-Reissner-Nordström solution in five dimensions

We start from the four-dimensional double Reissner-Nordström solution in the parameterization given recently by Manko in [25]. In our notation, the four-dimensional fields read:

$$\tilde{f} = \frac{A^2 - B^2 + C^2}{(A + B)^2}, \quad e^{2\tilde{\mu}} = \frac{A^2 - B^2 + C^2}{16\sigma_1^2\sigma_2^2(\nu + 2k)^2 r_1 r_2 r_3 r_4}, \quad \omega = -\frac{2C}{A + B}, \quad (3.12)$$

where:

$$\begin{aligned} A &= \sigma_1\sigma_2[\nu(r_1 + r_2)(r_3 + r_4) + 4k(r_1 r_2 + r_3 r_4)] - (\mu^2\nu - 2k^2)(r_1 - r_2)(r_3 - r_4), \\ B &= 2\sigma_1\sigma_2[(\nu M_1 + 2kM_2)(r_1 + r_2) + (\nu M_2 + 2kM_1)(r_3 + r_4)] \\ &\quad - 2\sigma_1[\nu\mu(Q_2 + \mu) + 2k(RM_2 + \mu Q_1 - \mu^2)](r_1 - r_2) \\ &\quad - 2\sigma_2[\nu\mu(Q_1 - \mu) - 2k(RM_1 - \mu Q_2 - \mu^2)](r_3 - r_4), \\ C &= 2\sigma_1\sigma_2\{[\nu(Q_1 - \mu) + 2k(Q_2 + \mu)](r_1 + r_2) + [\nu(Q_2 + \mu) + 2k(Q_1 - \mu)](r_3 + r_4)\} \\ &\quad - 2\sigma_1[\mu\nu M_2 + 2k(\mu M_1 + RQ_2 + \mu R)](r_1 - r_2) \\ &\quad - 2\sigma_2[\mu\nu M_1 + 2k(\mu M_2 - RQ_1 + \mu R)](r_3 - r_4), \end{aligned} \quad (3.13)$$

with constants:

$$\begin{aligned} \nu &= R^2 - \sigma_1^2 - \sigma_2^2 + 2\mu^2, & k &= M_1 M_2 - (Q_1 - \mu)(Q_2 + \mu), \\ \sigma_1^2 &= M_1^2 - Q_1^2 + 2\mu Q_1, & \sigma_2^2 &= M_2^2 - Q_2^2 - 2\mu Q_2, & \mu &= \frac{M_2 Q_1 - M_1 Q_2}{M_1 + M_2 + R}, \end{aligned} \quad (3.14)$$

while $r_i = \sqrt{\rho^2 + \zeta_i^2}$, for $i = 1 \dots 4$, with:

$$\zeta_1 = z - \frac{R}{2} - \sigma_2, \quad \zeta_2 = z - \frac{R}{2} + \sigma_2, \quad \zeta_3 = z + \frac{R}{2} - \sigma_1, \quad \zeta_4 = z + \frac{R}{2} + \sigma_1. \quad (3.15)$$

This solution is parameterized by five independent parameters and describes the superposition of two general Reissner-Nordström black holes, with masses $M_{1,2}$, charges $Q_{1,2}$ and R the coordinate distance separating them. For a detailed discussion of its properties we refer the reader to [25] and the references therein. We shall note here that in general the function $e^{2\tilde{\mu}}$ can be determined up to a constant and its precise numerical value has been fixed here by allowing the presence of conical singularities only in the portion in between the black holes along the φ axis. Consequently one has:

$$e^{2\tilde{\mu}}|_{\rho=0} = \left(\frac{\nu - 2k}{\nu + 2k}\right)^2, \quad (3.16)$$

for $-R/2 + \sigma_1 < z < R/2 - \sigma_2$ and $e^{2\tilde{\mu}}|_{\rho=0} = 1$ elsewhere.

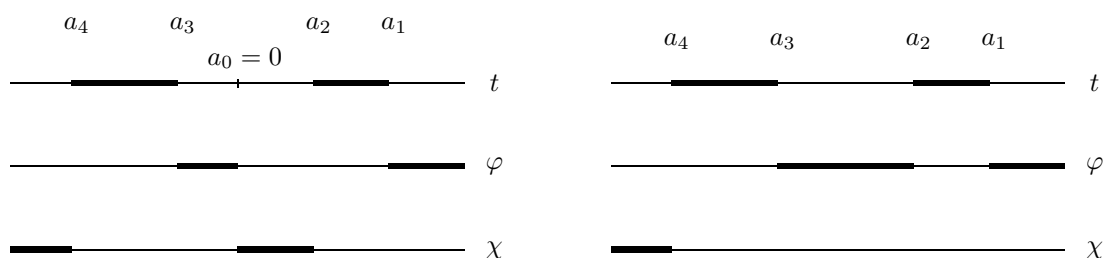


Figure 1: Rod structures of (a) the double-black hole system, and (b) the black Saturn.

Using the results from the previous section, the corresponding five-dimensional solution of the Einstein-Maxwell system reads:

$$\begin{aligned}
 ds_5^2 &= -\tilde{f}dt^2 + \tilde{f}^{-\frac{1}{2}} \left[e^{2h} d\chi^2 + e^{-2h} [e^{3\tilde{\mu}/2+2\gamma} (d\rho^2 + dz^2) + \rho^2 d\varphi^2] \right], \\
 A_t &= -\frac{\sqrt{3}C}{A+B}.
 \end{aligned} \tag{3.17}$$

So far the harmonic function h is still arbitrary. One can see that h 's presence can alter the rod structure of the final solution along the χ and φ directions. By carefully choosing the form of h , we can construct the appropriate rod structures to describe configurations involving black holes, black rings, or a combination of black holes and black rings. Once we pick a suitable h , γ is easily found by integrating (2.17). Let us illustrate this by considering three important cases.

3.2.1 Double-Reissner Nordström black holes

To describe a configuration of two black holes, it turns out that the appropriate choice for the harmonic function h is:

$$e^{2h} = \frac{\sqrt{(r_1 + \zeta_1)(r_2 + \zeta_2)(r_3 + \zeta_3)(r_4 + \zeta_4)}}{r_0 + \zeta_0}, \tag{3.18}$$

where we denote $r_i = \sqrt{\rho^2 + \zeta_i^2}$ and $\zeta_i = z - a_i$ for $i = 0 \dots 4$, a_i can be read from (3.15) and $a_0 = 0$. The corresponding rod structure of this solution is given in figure 1a).

Utilizing methods from [6] one can easily integrate (2.17) to find:

$$e^{2\gamma-2h} = \frac{1}{K_0 r_0} \frac{(Y_{01} Y_{02} Y_{03} Y_{04})^{\frac{1}{2}}}{(r_1 r_2 r_3 r_4 Y_{12} Y_{13} Y_{14} Y_{23} Y_{24} Y_{34})^{\frac{1}{4}}} \tag{3.19}$$

Here K_0 is an arbitrary constant whose value will be fixed later on, and $Y_{ij} = r_i r_j + \zeta_i \zeta_j + \rho^2$. The final solution describing the general double-Reissner-Nordström black hole is then given by (3.17) for this particular choice of the harmonic function h . The solution depends on five parameters and corresponds physically to the masses, charges of the two black holes and the distance between them.

Before we embark on a discussion of its physical properties, let us first consider the structure of the conical singularities along the axis. To define a conical singularity for a rotational axis with angle θ one computes the proper circumference C around the axis and its proper radius R and define:

$$\alpha = \frac{dC}{dR}|_{R=0} = \lim_{\rho \rightarrow 0} \frac{\sqrt{g_{\theta\theta}}\Delta\theta}{\int_0^\rho \sqrt{g_{\rho\rho}}d\rho} = \lim_{\rho \rightarrow 0} \frac{\partial_\rho \sqrt{g_{\theta\theta}}\Delta\theta}{\sqrt{g_{\rho\rho}}}, \quad (3.20)$$

where $\Delta\theta$ is the period of θ . The presence of a conical singularity is now expressed by means of:

$$\delta = 2\pi - \alpha, \quad (3.21)$$

such that $\delta > 0$ corresponds to a conical deficit (a ‘cosmic string’), while $\delta < 0$ corresponds to a conical excess (a ‘strut’).

Consider now the $ds_{\chi\rho}^2$ part of the metric. It turns out that conical singularities cannot be avoided and must be present either along $0 < z < R/2 - \sigma_2$ where:

$$\delta_\chi = 2\pi - \Delta\chi \sqrt{\frac{K_0}{\sqrt{8}}} \left(\frac{\nu + 2k}{\nu - 2k} \right)^{\frac{3}{4}} \left(\frac{16[(R + \sigma_2)^2 - \sigma_1^2][(R - \sigma_2)^2 - \sigma_1^2]}{(R^2 - 4\sigma_2^2)^2} \right)^{\frac{1}{4}},$$

or along $z < -R/2 - \sigma_1$, where we find:

$$\delta_\chi = 2\pi - \Delta\chi \sqrt{\frac{K_0}{\sqrt{8}}}.$$

If we set the period of χ be $\Delta\chi = 2\pi$ and choose to have a regular outer axis (for $z < -R/2 - \sigma_1$) we must set $K_0 = \sqrt{8}$. Similarly, for the $ds_{\varphi\rho}^2$ part of the metric, it turns out that conical singularities cannot be avoided and must be present either along $-R/2 + \sigma_1 < z < 0$, where:

$$\delta_\varphi = 2\pi - \Delta\varphi \sqrt{\frac{K_0}{\sqrt{8}}} \left(\frac{\nu + 2k}{\nu - 2k} \right)^{\frac{3}{4}} \left(\frac{16[(R + \sigma_2)^2 - \sigma_1^2][(R - \sigma_2)^2 - \sigma_1^2]}{(R^2 - 4\sigma_1^2)^2} \right)^{\frac{1}{4}},$$

or along $R/2 + \sigma_2 < z$, where we find:

$$\delta_\varphi = 2\pi - \Delta\varphi \sqrt{\frac{K_0}{\sqrt{8}}}.$$

Again, to have a regular outer axis we set $\Delta\varphi = 2\pi$ and $K_0 = \sqrt{8}$, in which case we still have a conical singularity on the axis in between the black holes. Therefore, if one demands the metric to be asymptotically flat with a regular outer axis, there will be conical defects between the black holes. Their presence is physically expected since our solution is static and therefore the conical defects should correspond to forces balancing the gravitational and electromagnetic forces in between the black holes.

One might wonder if there are some values of the parameters characterizing the solution for which the conical defects vanish. Our numerical investigation of this issue seems to imply a negative answer (the numerical methods used here are similar to those described in the black Saturn case). Although further work is clearly necessary, it appears that, similar to the four-dimensional case, equilibrium configurations require the presence of some unphysical features of the constituent black holes [25, 26, 29, 30].

Let us consider now some special limits of the above solution. First, in order to prove that this solution describes two Reissner-Nordström black holes, note that one can recover the individual black hole metric by pushing the other black hole to infinity. For example, to recover the metric for the second black hole (described by the parameters M_2 and Q_2) one has to first shift the z -coordinate $z \rightarrow z - R/2$ (i.e. positioning ones center on its horizon's rod) and then take the infinite separation limit $R \rightarrow \infty$. From the general expressions in (3.12), one notes that in this limit $\nu \sim R^2$, $\mu \sim 0$, $\sigma_i = \sqrt{M_i^2 - Q_i^2}$, for $i = 1 \dots 2$, $k = M_1 M_2 - Q_1 Q_2$ and:

$$A \sim 2\sigma_1\sigma_2 R^3(r_1 + r_2), \quad B \sim 4\sigma_1\sigma_2 R^3 M_2, \quad C \sim 4\sigma_1\sigma_2 R^3 Q_2. \quad (3.22)$$

Therefore the general solution (3.12) reduces to (3.1). Also, by taking this limit in the harmonic function h one obtains:

$$e^{2h} = 2[(r_2 + \zeta_2)(r_1 + \zeta_1)]^{\frac{1}{2}}, \quad e^{2\gamma-2h} = \frac{1}{[64r_1 r_2 Y_{12}]^{\frac{1}{4}}}. \quad (3.23)$$

Gathering all these results together and performing the coordinate transformation (3.6) one readily checks that the solution indeed reduces to the five-dimensional Reissner-Nordström black hole with conical singularities attached in the χ direction. Similarly, if one centers on the black hole on the left and pushes the other black hole to infinity one obtains the metric of a single Reissner-Nordström black hole with a conical singularity attached along the φ direction. As we shall see below, these conical singularities are unavoidable as they are inherited from the background geometry.

The uncharged case corresponds to setting $Q_1 = Q_2 = 0$ and noting that the four-dimensional seed solution (3.12) reduces in this case to the Israel-Khan solution [31] describing two neutral black holes, with the above choice of the harmonic function h one readily checks that one obtains the five-dimensional uncharged double-black hole solution constructed in [7].

The extremal charged limit of the above solution corresponds to taking the limits $M_1 = Q_1$ and $M_2 = Q_2$. This leads to $\sigma_1 = \sigma_2 = k = \mu = 0$ and, in consequence, $r_1 = r_2$ and $r_3 = r_4$. This extremality limit must be taken with care as one finds that the quantities A , B , C all vanish in this case. However, from the general expression of the metric functions in (3.12) one can nonetheless find the metric functions in the extremal case to be:

$$\tilde{f}_e = \left(1 + \frac{M_1}{r_3} + \frac{M_2}{r_1}\right)^{-2}, \quad e^{2\tilde{\mu}}|_e = 1. \quad (3.24)$$

Using these expressions in (3.17) one finally obtains the charged double-black hole solution previously found in [7]:

$$\begin{aligned}
 ds_{\text{ext}}^2 &= -H(r)^{-2}dt^2 + H(r) \left[\frac{(r_1 + \zeta_1)(r_3 + \zeta_3)}{r_0 + \zeta_0} d\chi^2 \right. \\
 &\quad \left. + \frac{Y_{01}Y_{03}}{4r_0r_1r_3Y_{13}} (d\rho^2 + dz^2) + \frac{(r_0 + \zeta_0)\rho^2 d\varphi^2}{(r_1 + \zeta_1)(r_3 + \zeta_3)} \right], \\
 A_t &= -\frac{\sqrt{3}}{2}H(r)^{-1}, \\
 H(r) &= 1 + \frac{M_1}{r_3} + \frac{M_2}{r_1}.
 \end{aligned} \tag{3.25}$$

Moreover, in absence of the black holes (we set $M_1 = M_2 = 0$) the background geometry is found to be:

$$ds_{\text{bkg}}^2 = -dt^2 + \frac{(r_1 + \zeta_1)(r_3 + \zeta_3)}{r_0 + \zeta_0} d\chi^2 + \frac{Y_{01}Y_{03}}{4r_0r_1r_3Y_{13}} (d\rho^2 + dz^2) + \frac{r_0 + \zeta_0}{(r_1 + \zeta_1)(r_3 + \zeta_3)} \rho^2 d\varphi^2.$$

As it is apparent from the rod structure in figure 1a), in absence of black holes, one recognizes the background to be the Euclidian form of the four-dimensional C-metric with an additional trivial time direction. It is clear now that one should always expect the presence of conical defects for any configuration of black holes in this background. In particular, as noted previously in [7], one finds unavoidable conical singularities even in the case of the extremal charged black hole.

3.2.2 The charged black Saturn solution

If one chooses to have a system consisting of a black ring with a black hole in its center (black Saturn), the appropriate harmonic function h is found to be:

$$e^{2h} = \sqrt{\frac{(r_1 + \zeta_1)(r_3 + \zeta_3)(r_4 + \zeta_4)}{(r_2 + \zeta_2)}}. \tag{3.26}$$

One can easily integrate (2.17) to find:

$$e^{2\gamma-2h} = \frac{1}{K_0} \left(\frac{Y_{12}Y_{23}Y_{24}}{r_1r_2r_3r_4Y_{13}Y_{14}Y_{34}} \right)^{\frac{1}{4}}, \tag{3.27}$$

where K_0 is a constant to be fixed when analyzing the conical singularities. The final solution (3.17) is again described by five dimensionful parameters, which correspond to the masses, charges of the black hole and black ring and the radius of the black ring. The rod structure of this solution corresponds to figure 1b). One can also consider various limits of the above solution as was performed for the double black hole solution in the previous section. In particular we checked that if one centers on the black hole horizon and sends $R \rightarrow \infty$ one obtains the metric of the single black hole. Since R now describes the radius of the black ring, the other limit, in which one centers on the black ring horizon and pushes $R \rightarrow \infty$ corresponds to making the radius of the ring very large and leads to a black string solution, as one can also infer from the rod structure of the black Saturn.

Turning now to a discussion of the conical defects, consider first the $ds_{\chi\rho}^2$ part of the metric. For the semi-infinite rod $z < -R/2 - \sigma_1$ along χ one finds:

$$\delta_\chi = 2\pi - \Delta\chi\sqrt{\frac{K_0}{2}}.$$

If we set the period of χ be $\Delta\chi = 2\pi$ and choose to have a regular outer axis (for $z < -R/2 - \sigma_1$) we must set $K_0 = 2$. Similarly, for the $ds_{\varphi\rho}^2$ part of the metric, it turns out that conical singularities cannot be avoided and must be present either along $-R/2 + \sigma_1 < z < R/2 - \sigma_2$ part of the axis, where:

$$\delta_\varphi = 2\pi - \Delta\varphi\sqrt{\frac{K_0}{2}} \left(\frac{\nu + 2k}{\nu - 2k}\right)^{\frac{3}{4}} \left(\frac{(R + \sigma_2)^2 - \sigma_1^2}{(R - \sigma_2)^2 - \sigma_1^2}\right)^{\frac{1}{4}},$$

or along $R/2 + \sigma_2 < z$, where we find:

$$\delta_\varphi = 2\pi - \Delta\varphi\sqrt{\frac{K_0}{2}}.$$

If one requires the outer axis $z > R/2 + \sigma_2$ be regular, one sets the period $\Delta\varphi = 2\pi$ and $K_0 = 2$. One then finds that there exists a conical defect in between the black ring and the black hole in its center. It is interesting to note that in order for the charged black Saturn system be in equilibrium one has to choose the parameters such that the following equilibrium condition is satisfied:

$$\left(\frac{\nu - 2k}{\nu + 2k}\right)^3 = \left(\frac{(R + \sigma_2)^2 - \sigma_1^2}{(R - \sigma_2)^2 - \sigma_1^2}\right). \tag{3.28}$$

Notice that this relation is satisfied for any value of R if one considers configurations of extremal objects, for which $k = \sigma_1 = \sigma_2 = 0$. However, the resulting configuration presents a naked singularity located on the black ring event horizon. This horizon singularity was already present in the case of a single extremally charged black ring.

We performed a numerical analysis of this equation searching for various values of the parameters describing non-extremal configurations. Although a systematic analysis of this issue is beyond the purpose of this paper, it turns out that this equation can be satisfied for families of non-extreme configurations. In the numerical analysis we fixed the length scale by taking $R = 1$, and looked for solutions with positive σ_i and positive ADM mass *i.e.* $M_1 + M_2 > 0$ (see the discussion in section 3.3 below). We further imposed the condition of non-overlapping horizons $R > \sigma_1 + \sigma_2$. In practice, we used an *F90* routine to evaluate equation (3.28) on a equidistant grid in (M_i, Q_i) , for some regions of the parameters space. The M_i, Q_i -points satisfying (3.28) within some given accuracy together with the conditions mentioned above, are then used as initial values for a standard equation solver. This provides solutions of the equation (3.28) with an accuracy of 10^{-14} . Our results indicate that configurations with real σ_i satisfying both the equilibrium and the non-overlapping horizons conditions exist. Moreover, such configurations exist for compact regions in the (M_i, Q_i) parameter space. An interesting case is provided by static black

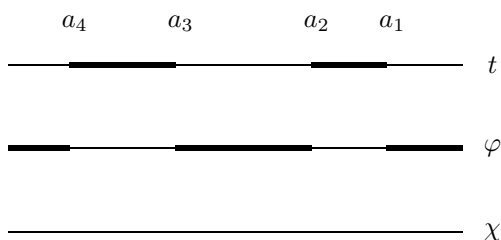


Figure 2: Rod structure of the double-black string system.

Saturns in equilibrium, which exhibit zero net electric charge as measured at infinity, i.e. $Q_1 + Q_2 = 0$, although there is a nonvanishing local charge density.

However, in all the solutions we found, the parameters M_1 and M_2 have opposite signs, although $M_1 + M_2 > 0$. The fact that these parameters have opposite signs does not necessarily imply that the equilibrium solutions have pathological properties. In fact, as we shall see later in section 3.3, the Komar masses of the individual constituents (as computed on the horizons) are proportional not to the parameters M_i but to the parameters σ_i , that is to the lengths of the rods determining the respective horizons. In our case, we find that the individual Komar masses are all positive!

However, our preliminary numerical results also suggest that for all the equilibrium solutions we found so far (with opposite signs of the mass parameters M_i) the denominator of f (i.e. $A + B$) seems to vanish for finite nonzero values of (ρ, z) and this signals the presence of naked curvature singularities outside the horizons since the Kretschmann scalar is proportional to $\frac{1}{(A+B)^6}$. Thus the existence of physically relevant static charged black Saturns remains an open problem.

3.2.3 Multi-black strings and non-extremal Majumdar-Papapetrou solutions

It is of interest to see the effects of “horizon corrections” solely, that is, if one does not use any rod-moving factors in h . For this purpose, let us consider now the following harmonic function h :

$$e^{2h} = \sqrt{\frac{(r_1 + \zeta_1)(r_3 + \zeta_3)}{(r_2 + \zeta_2)(r_4 + \zeta_4)}}. \quad (3.29)$$

One easily integrates (2.17) to find:

$$e^{2\gamma} = \frac{1}{K_0} \left(\frac{16Y_{12}Y_{14}Y_{23}Y_{34}}{r_1r_2r_3r_4Y_{13}Y_{24}} \right)^{\frac{1}{4}}, \quad (3.30)$$

where K_0 is a constant that can be fixed by demanding asymptotic flatness of the solution.

The rod structure of this solution is given in figure 3. One notices that there is no rod along the χ direction and, therefore, our solution should correspond to a configuration of

non-extremal charged black strings. To confirm this interpretation, let us again center on the horizon of one object and push the other to infinity. For convenience, let us center on the black hole on the right by shifting $z \rightarrow z - R/2$, and then taking the limit $R \rightarrow \infty$. We find that Manko's solution (3.12) reduces to the single Reissner-Nordström black hole solution, while:

$$e^{2h} = \sqrt{\frac{r_1 + \zeta_1}{r_2 + \zeta_2}}, \quad e^{2\gamma} = \frac{1}{K_0} \left(\frac{16Y_{12}}{r_1 r_2} \right)^{\frac{1}{4}}. \quad (3.31)$$

Performing now the coordinate transformations:

$$\rho = \sqrt{(r - m)^2 - \sigma^2} \sin \theta, \quad z = (r - m) \cos \theta, \quad (3.32)$$

and by appropriately choosing the value of K_0 one obtains the uniform black string solution:

$$ds^2 = -\frac{(r - m)^2 - \sigma^2}{r^2} dt^2 + \frac{r dr^2}{r - m - \sigma} + \frac{r d\chi^2}{r - m + \sigma} + r(r - m + \sigma)(d\theta^2 + \sin^2 \theta d\varphi^2),$$

$$A_t = \frac{\sqrt{3(m^2 - q^2)}}{r}, \quad (3.33)$$

as advertised.

Turning now to the discussion of the conical singularities, one finds that there is a conical singularity:

$$\delta_\varphi = 2\pi - \Delta\varphi \sqrt{\frac{K_0}{\sqrt{8}}}. \quad (3.34)$$

along the outer axis $z < -R/2 - \sigma_1$ or $z > R/2 + \sigma_2$, while:

$$\delta_\varphi = 2\pi - \Delta\varphi \sqrt{\frac{K_0}{\sqrt{8}}} \left(\left(\frac{\nu + 2k}{\nu - 2k} \right)^3 \frac{R^2 - (\sigma_1 + \sigma_2)^2}{R^2 - (\sigma_1 - \sigma_2)^2} \right)^{\frac{1}{4}}, \quad (3.35)$$

on the portion $-R/2 + \sigma_1 < z < R/2 - \sigma_2$ in between the black string horizons. We ensure regularity of the outer axis, by taking $\Delta\varphi = 2\pi$ and setting $K_0 = \sqrt{8}$. There will still be a conical singularity running in between the black strings. The equilibrium condition, for which this conical singularity disappears is given by:

$$\left(\frac{\nu - 2k}{\nu + 2k} \right)^3 = \frac{R^2 - (\sigma_1 + \sigma_2)^2}{R^2 - (\sigma_1 - \sigma_2)^2}. \quad (3.36)$$

However, when solving this equation numerically, we failed to find nonextremal solutions with $\delta_\varphi = 0$ also satisfying the physical conditions $M_1 + M_2 > 0$ and $\sigma_1 + \sigma_2 < R$.⁴

One clear way to satisfy (3.36) is to consider extremal objects for which $M_1 = Q_1$ and $M_2 = Q_2$. Again, this leads to $\sigma_1 = \sigma_2 = k = \mu = 0$ and, in consequence, $r_1 = r_2$ and $r_3 = r_4$. Using (3.24) and noticing that in the extremal limit one has $h = 0$, and therefore

⁴The numerical methods here were similar to those employed for the black Saturn case.

from (2.17) $\gamma = 0$, the final metric for the extremal double black string solution takes the simple form:

$$\begin{aligned}
 ds_{\text{MP}}^2 &= - \left(1 + \frac{M_1}{r_3} + \frac{M_2}{r_1}\right)^{-2} dt^2 + \left(1 + \frac{M_1}{r_3} + \frac{M_2}{r_1}\right) [d\chi^2 + d\rho^2 + dz^2 + \rho^2 d\varphi^2], \\
 A_t &= -\frac{\sqrt{3}}{2} \left(1 + \frac{M_1}{r_3} + \frac{M_2}{r_1}\right)^{-1}.
 \end{aligned}
 \tag{3.37}$$

The metric inside the square bracket describes Euclidian flat space and one recognizes the above solution as the particular case of the extremal Majumdar-Papapetrou double-black hole solution [17, 32].

3.3 Basic properties of the new solutions

All relevant quantities of the five-dimensional solutions can be expressed in terms of the parameters M_i, Q_i and R , which enter the four-dimensional seed solution. Explicitly, one finds that the Hawking temperature and event horizon area receive corrections that are fixed by the explicit form of the harmonic function h . Note that in the five-dimensional solution the horizons are still located at $\rho = 0, a_2 < z < a_1$ (upper black object) and $\rho = 0, a_4 < z < a_2$ (lower black object).

Near the black hole horizons, the leading order expressions of the functions that appear in the general line element (3.17) are given by:

$$e^{2h} \sim \sqrt{\rho}, \quad e^{\tilde{t}} \sim \rho, \quad \tilde{f} \sim \rho^2,
 \tag{3.38}$$

where the proportionality factors depend on z . Note that the following relations also hold near horizons, as implied by the (ρ, z) -component of Einstein's equations:

$$e^{4h-4\gamma} = (p^{(i)})^4 \rho + O(\rho^2).
 \tag{3.39}$$

The constant $p^{(i)}$, for each $i = 1, 2$ is fixed by the expression of h and takes different values for the upper and lower horizons.

The Hawking temperatures for each constituent of the five-dimensional solution can be computed either by evaluating the surface gravity or from the Euclidian section and it can be expressed as:

$$T_H^{(i)} = \frac{(\kappa^{(i)})^{3/4} p^{(i)}}{2\pi},
 \tag{3.40}$$

where $\kappa^{(i)}$ is the surface gravity of the i^{th} -black hole for the four-dimensional seed metric (3.12). The constants $\kappa^{(i)}$ are given by the relation (5.5) in the appendix, where they are expressed in terms of M_1, Q_1, M_2, Q_2 and R .

The horizon area of the i^{th} -black object is given by:

$$A_h^{(i)} = \frac{4\pi^2}{(\kappa^{(i)})^{3/4} p^{(i)}} \Delta z^{(i)},
 \tag{3.41}$$

for $i = 1 \dots 2$, where $\Delta z^{(1)} = a_1 - a_2 = 2\sigma_2$, $\Delta z^{(2)} = a_3 - a_4 = 2\sigma_1$. As usual, one identifies the entropy with one quarter of the event horizon area.

By using (3.17) and the relations (5.11) in the appendix, it is straightforward to show that the electric potential on the horizon of one of the black holes is:

$$\Phi^{(i)} = \frac{\sqrt{3}}{2} \left(\frac{M_i - \sigma_i}{Q_i} \right), \quad (3.42)$$

while the electric charges are evaluated according to:⁵

$$Q_e^{(i)} = \frac{1}{8\pi G} \int_S F_{\mu\nu} dS^{\mu\nu}, \quad (3.43)$$

and one obtains $Q_e^{(i)} = \frac{\sqrt{3}\pi}{G} Q_i$. If instead one computes this integral on the three-sphere at infinity enclosing both the black holes one finds the total electric charge $Q_e = Q_e^{(1)} + Q_e^{(2)}$.

To compute the ADM mass of the solutions, one performs the coordinate change $\rho = \frac{1}{2}r^2 \sin 2\theta$, $z = \frac{1}{2}r^2 \cos 2\theta$, and evaluates the expression of g_{tt} as $r \rightarrow \infty$. One finds that the total mass is given by:

$$M_{\text{ADM}} = \frac{3\pi}{2G} (M_1 + M_2). \quad (3.44)$$

One can also evaluate the Komar mass of an individual black object, by using the definition:

$$M = -\frac{1}{16\pi G} \frac{3}{2} \int_S \alpha, \quad (3.45)$$

where S is the boundary of any spacelike hypersurface and:

$$\alpha_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma\tau} \nabla^\sigma \xi^\tau, \quad (3.46)$$

with the Killing vector $\xi = \partial/\partial t$. This relation measures the mass contained in S , and therefore the horizon mass M_H is obtained by performing the above integration at the horizon. If we take S to be the three-sphere at infinity enclosing both horizons instead, then (3.45) gives the total mass of the system, which coincides with the ADM mass. A straightforward computation leads to:

$$M_{\text{Komar}}^{(1)} = \frac{3}{8\pi G} \sigma_2 \Delta\varphi \Delta\chi, \quad M_{\text{Komar}}^{(2)} = \frac{3}{8\pi G} \sigma_1 \Delta\varphi \Delta\chi, \quad (3.47)$$

while

$$M = M_{\text{Komar}}^{(1)} + M_{\text{Komar}}^{(2)} - \frac{1}{16\pi G} \frac{3}{2} \int R_t^t \sqrt{-g} dV. \quad (3.48)$$

However, since Einstein's equations imply $R_t^t = \frac{F_{\mu t}^2}{3}$, one arrives at the following five-dimensional Smarr formula [28]:

$$M = \mathcal{M}^{(1)} + \mathcal{M}^{(2)},$$

where for each constituent one has:⁶

$$\frac{2}{3} \mathcal{M}^{(i)} = T_H^{(i)} S^{(i)} + \frac{2}{3} \Phi_H^{(i)} Q_e^{(i)},$$

⁵Here the integration is performed at the horizon.

⁶This relation follows from the four-dimensional Smarr formula (5.12).

where $\mathcal{M}^{(i)} = \frac{3\pi}{2G}M_i$. Thus one can regard $\mathcal{M}^{(i)}$ as the individual mass of each black object, containing an electromagnetic contribution apart from the Komar part! Notice that there is no compelling reason to impose $\mathcal{M}^{(i)} > 0$ as long as the total mass M measured at infinity is still positive. Moreover, the Komar mass of each constituent is positive, since it is proportional to the length of the rod determining each horizon. Finally, let us notice that for $|Q_i| = M_i$ the extremality condition is indeed satisfied [46]:

$$\frac{\mathcal{M}^{(i)}}{|Q_e^{(i)}|} = \frac{\sqrt{3}}{2}. \quad (3.49)$$

The situation is slightly different for the double black string solutions. Since the background approached at infinity is $\mathcal{M}^4 \times S^1$, the black strings have also a nonzero tension, which is the charge associated with the Killing vector $\partial/\partial\chi$. To find the ADM mass and tension of the uniform black strings, we consider the asymptotics g_{tt} , $g_{\chi\chi}$ in a coordinate system with $\rho = r \sin \theta$, $z = r \cos \theta$:

$$g_{tt} = -1 + \frac{c_t}{r} + O(1/r^2), \quad g_{\chi\chi} = 1 + \frac{c_\chi}{r} + O(1/r^2), \quad (3.50)$$

with $c_t = 2(M_1 + M_2)$, $c_\chi = M_1 + M_2 - \sigma_1 - \sigma_2$. Thus [47]

$$M_{\text{ADM}} = \frac{LV_2}{16\pi G}(2c_t - c_\chi), \quad \mathcal{T}_{\text{ADM}} = \frac{V_2}{16\pi G}(c_t - 2c_\chi), \quad (3.51)$$

where $V_2 = 4\pi$ and $L = \Delta\chi$, which agree with the Komar mass and tension M , \mathcal{T} . The black strings satisfy the Smarr relation:

$$\frac{1}{3}(2M - L\mathcal{T}) = T_H^{(1)}S^{(1)} + T_H^{(2)}S^{(2)} + \frac{2}{3}(\Phi_H^{(1)}Q_e^{(1)} + \Phi_H^{(2)}Q_e^{(2)}). \quad (3.52)$$

The above relations allow a discussion of the basic physical properties of the solutions we found. Take for instance the double black hole solution discussed in section 3.2.1. The first black hole horizon is located at $\rho = 0$ for $a_2 \leq z \leq a_1$ and the metric on a spatial cross-section of the horizon can be written as:

$$ds_{BH^1}^2 = \sqrt{\frac{\zeta_2\zeta_3\zeta_4}{|\zeta_1|}} \frac{1}{z\sqrt{F^{(1)}(z)}} d\chi^2 + \frac{F^{(1)}(z)}{(\kappa^{(1)})^{3/2}(p^{(1)})^2} dz^2 + \sqrt{\frac{|\zeta_1|}{\zeta_2\zeta_3\zeta_4}} \frac{z}{\sqrt{F^{(1)}(z)}} d\varphi^2, \quad (3.53)$$

where, as implied by (3.39):

$$p^{(1)} = \left(\frac{2K_0^2\sigma_2(R - \sigma_1 + \sigma_2)(R + \sigma_1 + \sigma_2)}{(R + 2\sigma_2)^2} \right)^{1/4}. \quad (3.54)$$

The second black hole horizon is located at $\rho = 0$ for $a_4 \leq z \leq a_3$. The metric on a spatial cross-section of the horizon is:

$$ds_{BH^2}^2 = \sqrt{\frac{\zeta_4}{|\zeta_1\zeta_2\zeta_3|}} \frac{z}{\sqrt{F^{(2)}(z)}} d\chi^2 + \frac{F^{(2)}(z)}{(\kappa^{(2)})^{3/2}(p^{(2)})^2} dz^2 + \frac{1}{z} \sqrt{\frac{|\zeta_1\zeta_2\zeta_3|}{\zeta_4}} \frac{1}{\sqrt{F^{(2)}(z)}} d\varphi^2, \quad (3.55)$$

where

$$p^{(2)} = \left(\frac{2K_0^2 \sigma_2 (R + \sigma_1 - \sigma_2)(R + \sigma_1 + \sigma_2)}{(R + 2\sigma_1)^2} \right)^{1/4}. \quad (3.56)$$

One can easily see that the topology of the horizon is S^3 in both cases, as expected from the rod diagram in figure 1a).

Turning now to the charged black Saturn solution derived in section 3.2.2, the black ring horizon is located at $\rho = 0$ for $a_2 \leq z \leq a_1$. The metric of the spatial cross section of the black ring horizon reads:

$$ds_{\text{BR}}^2 = \sqrt{\frac{\zeta_3 \zeta_4}{|\zeta_1| \zeta_2}} \frac{1}{\sqrt{F^{(1)}(z)}} d\chi^2 + \frac{F^{(1)}(z)}{(\kappa^{(1)})^{3/2} (p^{(1)})^2} dz^2 + \sqrt{\frac{|\zeta_1| \zeta_2}{\zeta_3 \zeta_4}} \frac{1}{\sqrt{F^{(1)}(z)}} d\varphi^2, \quad (3.57)$$

where we denoted:

$$p^{(1)} = \left(\frac{K_0^2 (R + \sigma_2 - \sigma_1)(R + \sigma_1 + \sigma_2)}{4\sigma_2} \right)^{1/4}. \quad (3.58)$$

Along the black ring horizon, the orbits of φ shrink to zero at $z = a_1$ and $z = a_2$, while the orbits of χ do not shrink to zero anywhere. Thus the topology of the horizon is $S^2 \times S^1$ as expected from the rod diagram in figure 1b). However, the black hole horizon is located at $\rho = 0$ for $a_4 \leq z \leq a_3$. The metric on the spatial cross-section of the black hole horizon is in this case given by:

$$ds_{\text{BH}}^2 = \sqrt{\frac{|\zeta_2| \zeta_4}{|\zeta_1| \zeta_3}} \frac{1}{\sqrt{F^{(2)}(z)}} d\chi^2 + \frac{F^{(2)}(z)}{(\kappa^{(2)})^{3/2} (p^{(2)})^2} dz^2 + \sqrt{\frac{|\zeta_1| \zeta_3}{|\zeta_2| \zeta_4}} \frac{1}{\sqrt{F^{(2)}(z)}} d\varphi^2, \quad (3.59)$$

where:

$$p^{(2)} = \left(\frac{K_0^2 \sigma_1 (R + \sigma_1 + \sigma_2)}{(R + \sigma_1 - \sigma_2)} \right)^{1/4}. \quad (3.60)$$

The orbits of φ shrink to zero at $z = a_3$, while the orbits of χ shrink to zero at $z = a_4$. Thus the topology of the horizon is indeed S^3 and it corresponds to a black hole.

A similar computation can be performed for the two-black string solution presented in section 3.2.3. The metric of the spatial cross section of black strings horizons reads:

$$ds_{BS_1}^2 = \sqrt{\frac{\zeta_3}{|\zeta_1| \zeta_2 \zeta_4}} \frac{1}{2\sqrt{F^{(1)}(z)}} d\chi^2 + \frac{F^{(1)}(z)}{(\kappa^{(1)})^{3/2} (p^{(1)})^2} dz^2 + \sqrt{\frac{|\zeta_1| \zeta_2 \zeta_4}{\zeta_3}} \frac{2}{\sqrt{F^{(1)}(z)}} d\varphi^2, \quad (3.61)$$

for the first black string horizon, respectively:

$$ds_{BS_2}^2 = \sqrt{\frac{|\zeta_2|}{\zeta_1 \zeta_3 \zeta_4}} \frac{1}{2\sqrt{F^{(2)}(z)}} d\chi^2 + \frac{F^{(2)}(z)}{(\kappa^{(1)})^{3/2} (p^{(2)})^2} dz^2 + \sqrt{\frac{\zeta_1 \zeta_3 \zeta_4}{|\zeta_2|}} \frac{2}{\sqrt{F^{(2)}(z)}} d\varphi^2, \quad (3.62)$$

for the second black string horizon, where we denoted:

$$p^{(1)} = \left(\frac{K_0^2 (R + \sigma_2 - \sigma_1)}{32\sigma_2 (R + \sigma_1 + \sigma_2)} \right)^{1/4}, \quad p^{(2)} = \left(\frac{K_0^2 (R + \sigma_1 - \sigma_2)}{32\sigma_1 (R + \sigma_1 + \sigma_2)} \right)^{1/4}. \quad (3.63)$$

One can see that the orbits of φ shrink to zero at $z = a_1$ and $z = a_2$ for the first black string, and at $z = a_3$ and $z = a_4$ for the second black string, while the orbits of χ do not shrink to zero anywhere. Thus the topology of the horizon is indeed $S^2 \times S^1$ as expected from the rod diagram in figure 3.

4. Conclusions

In this paper, by using a novel solution generation technique we were able to construct the general non-extremally charged multi-black hole solutions in five dimensions. As opposed to other solution-generating methods used previously in the literature to construct five-dimensional solutions [34–37] our method lifts a four-dimensional static charged solution of Einstein-Maxwell field equations to a solution in the more general Einstein-Maxwell-Dilaton theory in five dimensions. While the fields of the general EMD solution can be read in each case from (3.17), in discussing the generated solutions we focused for simplicity on Einstein-Maxwell theory, for which the coupling constant $\alpha = 0$ in the general solution (3.17) vanishes.

For simplicity we restricted our attention to configurations consisting of only two constituents. In four dimensions there exists a general solution describing a static configuration of two Reissner-Nordström black holes, which was recently cast into a simpler form in [25, 26]. When lifted to five dimensions, we found solutions describing general static configurations of charged black objects. These solutions include as particular cases the charged black Saturn, the double non-extremal Reissner-Nordström solution and the double black string solution, whose extremal limit we found to be precisely of the five dimensional Majumdar-Papapetrou type solution. Even though we found static configurations of non-extremal black holes/rings/strings, our numerical results indicate the presence of naked curvature singularities for finite values of ρ (outside the horizons at $\rho = 0$) for all charged black Saturn solutions satisfying the equilibrium condition. We are skeptical that any equilibrium charged black saturn solutions exist without angular momenta.

We also note that for suitable choices of the harmonic function h the generated solutions can describe configurations of two black rings (orthogonal or concentric). These double-ring solutions would correspond to the charged versions of the static di-ring solution and the bicycling black ring system [5, 38–43]. For example, in the di-ring case one takes:

$$e^{2h} = (r - \zeta_0) \sqrt{\frac{(r_1 + \zeta_1)(r_3 + \zeta_3)}{(r_2 + \zeta_2)(r_4 + \zeta_4)}}, \tag{4.1}$$

where $\zeta_0 = z - \sigma_0$, with $\sigma_0 > \sigma_2$, while for the charged bi-ring system one takes:

$$e^{2h} = (r + \zeta_0) \sqrt{\frac{(r_1 + \zeta_1)(r_4 + \zeta_4)}{(r_2 + \zeta_2)(r_3 + \zeta_3)}}, \tag{4.2}$$

where $\zeta_0 = z$. In both cases it is trivial to integrate (2.17) to find explicitly the factor $e^{2\gamma}$, which enters the general solution.

One should also remark at this point that instead of using the four-dimensional double-Reissner Nordström as the seed metric, we could have used the more general solution describing general configurations of N Reissner-Nordström black holes given in [33]. For such a configuration the choice of the harmonic function h in our solution-generating technique is easily inferred from the rod diagram of the solution one wishes to describe. One also has to add the correction factor for each black object horizon as described in section 2.

As a general remark, since all the five dimensional solutions constructed using the double-Reissner-Nordström four dimensional solution as the seed metric are sufficiently complicated, it is generally a very difficult task to check algebraically that they satisfy the Einstein-Maxwell equations. However, the correctness of our general solutions was confirmed via numerical methods and we explicitly verified that our solutions satisfy the field equations for several sets of constants (M_i, Q_i, R) .

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5. The double-Reissner-Nordström solution in four dimensions

We briefly present the basic properties of the double-Reissner-Nordström solution in four dimensions. In the parameterization given recently by Manko in [25], the four-dimensional quantities that appear in (3.12) are:

$$A = \sum_{1 \leq i < j \leq 4} a_{ij} r_i r_j, \quad B = \sum_{i=1}^4 b_i r_i, \quad C = \sum_{i=1}^4 c_i r_i, \quad (5.1)$$

where we defined:

$$\begin{aligned} a_{12} &= a_{34} = 4k\sigma_1\sigma_2, \\ a_{13} &= a_{24} = 2k^2 - \nu(\mu^2 - \sigma_1\sigma_2), \\ a_{14} &= a_{23} = -2k^2 + \nu(\mu^2 + \sigma_1\sigma_2), \\ b_1 &= 2\sigma_1[\nu(-\mu(\mu + Q_2) + M_1\sigma_2) + 2k(\mu(\mu - Q_1) + M_2(\sigma_2 - R))], \\ b_2 &= 2\sigma_1[\nu(\mu(\mu + Q_2) + M_1\sigma_2) + 2k(\mu(-\mu + Q_1) + M_2(\sigma_2 + R))], \\ b_3 &= 2\sigma_2[\nu(\mu(\mu - Q_1) + M_2\sigma_1) - 2k(\mu(\mu + Q_2) - M_1(\sigma_1 + R))], \\ b_4 &= 2\sigma_2[\nu(\mu(-\mu + Q_1) + M_2\sigma_1) + 2k(\mu(\mu + Q_2) + M_1(\sigma_1 - R))], \\ c_1 &= -2\sigma_1[2k(\mu M_1 + (\mu + Q_2)(R - \sigma_2)) + \nu(M_2\mu + \sigma_2(\mu - Q_1))], \\ c_2 &= 2\sigma_1[2k(\mu M_1 + (\mu + Q_2)(R + \sigma_2)) + \nu(M_2\mu + \sigma_2(-\mu + Q_1))], \\ c_3 &= -2\sigma_2[2k(\mu M_2 + (\mu - Q_1)(R + \sigma_1)) + \nu(M_1\mu - \sigma_1(\mu + Q_2))], \\ c_4 &= 2\sigma_2[2k(\mu M_2 + (\mu - Q_1)(R - \sigma_1)) + \nu(M_1\mu + \sigma_1(\mu + Q_2))]. \end{aligned} \quad (5.2)$$

Depending on the values of M_i, Q_i, R , this solution describes two non-extremal black holes, two naked singularities or a black hole-naked singularity. The equilibrium is possible only

in this last case. However, we shall consider only the case of real σ_i , corresponding to a configuration of two interacting black holes. The black hole event horizons are located at $\rho = 0$ and $a_2 < z < a_1$ (the first black hole) and $a_4 < z < a_3$ (the second black hole).

It is also useful to present the approximate expressions of the basic pieces A, B, C near one of the horizons:

$$\begin{aligned}
 A(\rho, z) &= A_0(z) + A_2(z)\rho^2 + O(\rho^4), \\
 &\quad \text{with } A_0 = \sum_{1 \leq i < j \leq 4} a_{ij} |\zeta_i \zeta_j|, \quad A_2 = \frac{1}{2} \sum_{1 \leq i < j \leq 4} a_{ij} \frac{(\zeta_i^2 + \zeta_j^2)}{|\zeta_i \zeta_j|}, \\
 B(\rho, z) &= B_0(z) + B_2(z)\rho^2 + O(\rho^4), \\
 &\quad \text{with } B_0 = \sum_{i=1}^4 b_i |\zeta_i|, \quad B_2 = \frac{1}{2} \sum_{i=1}^4 b_i \frac{1}{|\zeta_i|}, \\
 C(\rho, z) &= C_0(z) + C_2(z)\rho^2 + O(\rho^4), \\
 &\quad \text{with } C_0 = \sum_{i=1}^4 c_i |\zeta_i|, \quad C_2 = \frac{1}{2} \sum_{i=1}^4 c_i \frac{1}{|\zeta_i|}, \tag{5.3}
 \end{aligned}$$

which implies the following near horizon expression for the function f :

$$f(\rho, z) = F(z)\rho^2 + O(\rho^4), \quad \text{with } F(z) = 2 \frac{A_0 A_2 - B_0 B_2 + C_0 C_2}{(A_0 + B_0)^2}.$$

We emphasize that $F(z)$ has a different expression for each horizon, the corresponding function being labeled as $F^{(i)}(z)$, $i = 1, 2$. One can easily see that as $z \rightarrow a_i$ (near the ends of the rods), $F(z) \sim 1/|\zeta_i|$.

The two horizons have different Hawking temperatures, which are given by:

$$T_H^{(i)} = \frac{\kappa^{(i)}}{2\pi} \tag{5.4}$$

with the surface gravities:

$$\kappa^{(i)} = \frac{k_0^{(i)}}{t_0^{(i)}} \sqrt{t_A^{(i)} - t_B^{(i)} + t_C^{(i)}} \tag{5.5}$$

where:

$$\begin{aligned}
 t_A^{(k)} &= -a_1 a_2 a_3 a_4 \left(\sum_{1 \leq i < j \leq 4} a_{ij} a_i a_j \epsilon_{ij}^{(k)} \right) \left(\sum_{1 \leq l < m \leq 4} a_{lm} \epsilon_{lm}^{(k)} \frac{(a_l^2 + a_m^2)}{a_l a_m} \right), \\
 t_B^{(k)} &= -a_1 a_2 a_3 a_4 \left(\sum_{i=1}^4 \frac{b_i}{a_i} \epsilon_{ii}^{(k)} \right) \left(\sum_{j=1}^4 a_j b_j \epsilon_{jj}^{(k)} \right), \\
 t_C^{(k)} &= -a_1 a_2 a_3 a_4 \left(\sum_{i=1}^4 \frac{c_i}{a_i} \epsilon_{ii}^{(k)} \right) \left(\sum_{j=1}^4 c_j b_j \epsilon_{jj}^{(k)} \right), \\
 t_0^{(k)} &= \left(\sum_{1 \leq i < j \leq 4} a_{ij} a_i a_j \epsilon_{ij}^{(k)} + \sum_{i=1}^4 a_i b_i \epsilon_{ii}^{(k)} \right)^2. \tag{5.6}
 \end{aligned}$$

The symbol $\epsilon_{ij}^{(k)}$ is defined such that for the upper black hole one takes:

$$-\epsilon_{11}^{(1)} = \epsilon_{22}^{(1)} = \epsilon_{33}^{(1)} = \epsilon_{44}^{(1)} = 1, \quad \epsilon_{12}^{(1)} = \epsilon_{13}^{(1)} = \epsilon_{14}^{(1)} = -\epsilon_{23}^{(1)} = -\epsilon_{24}^{(1)} = -\epsilon_{34}^{(1)} = 1, \quad (5.7)$$

while for the lower black hole one takes:

$$\epsilon_{11}^{(2)} = \epsilon_{22}^{(2)} = \epsilon_{33}^{(2)} = -\epsilon_{44}^{(2)} = 1, \quad \epsilon_{12}^{(2)} = \epsilon_{13}^{(2)} = \epsilon_{23}^{(2)} = -\epsilon_{14}^{(2)} = -\epsilon_{24}^{(2)} = -\epsilon_{34}^{(2)} = 1. \quad (5.8)$$

The event horizon area of each black hole is given by:

$$A_h^{(i)} = \frac{2\pi}{\kappa^{(i)}} \Delta z^{(i)}, \quad (5.9)$$

where we denoted $\Delta z^{(1)} = a_1 - a_2 = 2\sigma_1$ and $\Delta z^{(2)} = a_3 - a_4 = 2\sigma_2$.

The electric charges of the black holes are Q_1 for the upper one and Q_2 for the lower black hole [25]. The electrostatic potential on the black hole horizons is constant and can be generally expressed as:

$$V^{(k)} = -\frac{2 \sum_{i=1}^4 a_i c_i \epsilon_{ii}^{(k)}}{\sum_{1 \leq i < j \leq 4} a_{ij} a_i a_j \epsilon_{ij}^{(k)} + \sum_{i=1}^4 a_i b_i \epsilon_{ii}^{(k)}} \quad (5.10)$$

Unfortunately, after replacing a_{ij}, a_i, b_i, c_i in the expressions of $T_H^{(i)}, A_h^{(i)}$ the resulting formulae in terms of M_i, Q_i, R cannot be further simplified. However, $V^{(k)}$ can be re-expressed in the simple form:

$$V^{(i)} = \frac{M_i - \sigma_i}{Q_i}. \quad (5.11)$$

The total ADM mass of the system is $M = M_1 + M_2$. The following Smarr relation also holds in four-dimensions [28]:

$$M_i = \frac{1}{2} T_H^{(i)} A_h^{(i)} + Q_i V^{(i)}. \quad (5.12)$$

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